Biring and plethory structures on integer-valued polynomial rings

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Abstract

Let A and B be commutative rings with identity. An A-B-*biring* is an A-algebra S together with the structure on S of a B-algebra object in the opposite category of the category of A-algebras; equivalently, an A-B-biring is an A-algebra S together with a lift of the functor $\operatorname{Hom}_A(S, -)$ from A-algebras to sets to a functor from A-algebras to B-algebras. An A-*plethory* is a monoid object in the monoidal category, equipped with the composition product, of A-A-birings. We show that $\operatorname{Int}(D)$ has such a structure if $D = A$ is a domain such that the natural D-algebra homomorphism $\theta_n : \bigotimes_{D}{}_{i=1}^{n} \operatorname{Int}(D) \longrightarrow \operatorname{Int}(D^n)$ is an isomorphism for $n = 2$ and injective for $n \leq 4$. This holds in particular if θ_n is an isomorphism for all n, which in turn holds, for example, if D is a Krull domain or more generally a TV PVMD. In these cases we also examine properties of the functor $\operatorname{Hom}_D(\operatorname{Int}(D), -)$ from D-algebras to D-algebras.

1 Introduction

Throughout this paper all rings and algebras are assumed commutative with identity. For any integral domain D with quotient field K, any set $\mathbf{X}$, and any subset $\mathbf{E}$ of $K^{\mathbf{X}}$, the ring of *integer-valued polynomials on $\mathbf{E}$ over D* is the subring

$$\operatorname{Int}(\mathbf{E}, D) = \{f(\mathbf{X}) \in K[\mathbf{X}] : f(\mathbf{E}) \subseteq D\}$$

of the polynomial ring $K[\mathbf{X}]$. In other words, $\operatorname{Int}(\mathbf{E}, D)$ is the pullback of the direct product $D^{\mathbf{E}}$ along the K-algebra homomorphism $K[\mathbf{X}] \longrightarrow K^{\mathbf{E}}$ acting by $f \longmapsto (f(\underline{a}))_{\underline{a} \in \mathbf{E}}$. One writes $\operatorname{Int}(D^{\mathbf{X}}) = \operatorname{Int}(D^{\mathbf{X}}, D)$ and $\operatorname{Int}(D) = \operatorname{Int}(D, D)$. One also writes $\operatorname{Int}(D^n) = \operatorname{Int}(D^{\mathbf{X}})$ if $\mathbf{X}$ is a set of cardinality n.

Much of the theory of integer-valued polynomial rings developed in attempts to generalize results known about $\operatorname{Int}(\mathbb{Z})$ to $\operatorname{Int}(D)$. This paper is concerned with

finding such a generalization of a particular result about $\text{Int}(\mathbb{Z})$. To state this result we need a few definitions.

A ring A is said to be *binomial* if A is $\mathbb{Z}$-torsion-free and $\frac{a(a-1)(a-2)\cdots(a-n+1)}{n!} \in A \otimes_{\mathbb{Z}} \mathbb{Q}$ lies in A for all $a \in A$ and all positive integers n. For any set $\mathbf{X}$ the ring $\text{Int}(\mathbb{Z}^{\mathbf{X}})$ is the free binomial ring generated by $\mathbf{X}$, and a $\mathbb{Z}$-torsion-free ring A is binomial if and only if, for every $a \in A$, there exists a ring homomorphism $\text{Int}(\mathbb{Z}) \longrightarrow A$ sending X to a [4]. By the category of binomial rings we will mean the full subcategory of the category of rings whose objects are the binomial rings. For any ring A we let $\Lambda(A)$ denote the *universal λ-ring over A*. As an abelian group, the ring $\Lambda(A)$ is the group $1 + TA[[T]]$ of formal power series in A with constant coefficient equal to 1. The ring $\Lambda(A)$ is a λ-ring whose Adams operations are the well-known Frobenius endomorphisms of $\Lambda(A)$. See [10] for definitions of $\Lambda(A)$, *Frobenius endomorphism*, λ-ring, and *Adams operation*.

By [4, Theorem 9.1], a binomial ring is equivalently a λ-ring A whose Adams operations are all the identity on A. For any ring A, let $\text{Bin}(A)$ denote the set all elements of $\Lambda(A)$ that are fixed by all of the Adams operations on $\Lambda(A)$. The ring $\text{Bin}(A)$ is functorial in A, and the association $A \longmapsto \text{Bin}(A)$ defines a functor from the category of rings to the category of binomial rings. By [1, Section 46] and [4, Theorem 9.1] we have the following.

Proposition 1.1. *The functor* Bin *from rings to binomial rings is left-represented by* $\text{Int}(\mathbb{Z})$ *and is a right adjoint for the inclusion from binomial rings to rings.*

Our motivating problem is to generalize the above result to $\text{Int}(D)$ for further domains D. More specifically, we are interested in the following.

Problem 1.2. *Determine all domains D for which $\text{Int}(D)$ left-represents a right adjoint for the inclusion from $\mathcal{C}$ to the category of D-algebras for some full subcategory $\mathcal{C}$ of the category of D-algebras.*

In particular, if D is such a domain, then the functor $\text{Hom}_D(\text{Int}(D), -)$ from D-algebras to sets must lift to a functor from D-algebras to D-algebras in $\mathcal{C}$. If $D = \mathbb{Z}$, then by Proposition 1.1 this holds for the category $\mathcal{C}$ of binomial rings. Given a domain D, a natural candidate for the category $\mathcal{C}$ is the category of D-torsion-free "weakly polynomially complete" D-algebras, where a D-algebra A is said to be *weakly polynomially complete*, or *WPC*, if for every $a \in A$ there exists a D-algebra homomorphism $\text{Int}(D) \longrightarrow A$ sending X to a. A binomial ring is equivalently a $\mathbb{Z}$-torsion-free WPC $\mathbb{Z}$-algebra, and for any domain D the D-algebra $\text{Int}(D)$ is itself WPC. Our goal, then, is to construct a right adjoint for the inclusion from the category of D-torsion-free WPC D-algebras to the category of D-algebras that is left-represented by $\text{Int}(D)$. In our efforts to do so we found it necessary to utilize the notions of a *biring* and a *plethory*.

Let A and B be rings. An A-B-*biring* is an A-algebra S together with the structure on S of a B-algebra object in the opposite category of the category of A-algebras. Thus an A-B-biring is an A-algebra S equipped with two *binary*

co-operations $S \longmapsto S \otimes_A S$, called *co-addition* and *co-multiplication* (both of which are A-algebra homomorphisms), along with a *co-B-linear structure* $B \longrightarrow \operatorname{Hom}_A(S, A)$, satisfying laws dual to those defining the A-algebras. By Yoneda's lemma, an A-B-biring is equivalently an A-algebra S together with a lift of the covariant functor $\operatorname{Hom}_A(S, -)$ it represents to a functor from the category of A-algebras to the category of B-algebras. (See any of [1, 2, 9] for the details.) For example, the polynomial ring $A[X]$ is an A-A-biring as it represents the identity functor from the category of A-algebras to itself. Co-addition acts by $X \longmapsto X \otimes 1 + 1 \otimes X$, co-multiplication by $X \longmapsto X \otimes X$, and the co-linear structure by $a \longmapsto (f \longmapsto f(a))$.

The following proposition is clear.

Proposition 1.3. *Let D be an integral domain.*

1. *The existence of a D-D-biring structure on $\operatorname{Int}(D)$ is equivalent to the existence of a lift of the functor $\operatorname{Hom}_D(\operatorname{Int}(D), -)$ from D-algebras to sets to a functor from D-algebras to D-algebras.*

2. *A D-D-biring structure on $\operatorname{Int}(D)$ is compatible with the D-D-biring structure on $D[X]$, that is, the inclusion $D[X] \longrightarrow \operatorname{Int}(D)$ is a homomorphism of D-D-birings, if and only if the natural map $\operatorname{Hom}_D(\operatorname{Int}(D), A) \longrightarrow A$ given by $\varphi \longmapsto \varphi(X)$ is a D-algebra homomorphism for every D-algebra A.*

Consequently, any solution to Problem 1.2 would yield conditions on integral domains D under which the D-algebra $\operatorname{Int}(D)$ has a D-D-biring structure. Regarding the latter problem we prove the following.

Theorem 1.4. *Let D be an integral domain such that $\operatorname{Int}(D)$ is flat over D, or more generally such that the n-th tensor power $\operatorname{Int}(D)^{\otimes n}$ of $\operatorname{Int}(D)$ over D is D-torsion-free for $n \leq 4$. Then the domain $\operatorname{Int}(D)$ has a (necessarily unique) D-D-biring structure that is compatible with the D-D-biring structure on $D[X]$ if and only if for every $f \in \operatorname{Int}(D)$ the polynomials $f(X + Y)$ and $f(XY)$ both can be expressed as sums of polynomials of the form $g(X)h(Y)$ for $g, h \in \operatorname{Int}(D)$.*

We also prove that the hypotheses of the above theorem hold for a substantial class of domains that includes all Krull domains and more generally all TV PVMDs. See [8] for the definition of PVMDs and TV PVMDs.

By [2, Proposition 1.4], for any A-B-biring S, the lifted functor $\operatorname{Hom}_A(S, -)$ from A-algebras to B-algebras has a left adjoint, denoted $S \odot_A -$. In analogy with the tensor product, the A-algebra $S \odot_A R$ for any B-algebra R is the A-algebra generated by the symbols $s \odot r$ for all $s \in S$ and $r \in R$, subject to the relations [2, 1.3.1–2]. If S and T are A-A-birings, then so is $S \odot_A T$, and the category of A-A-birings equipped with the operation $\odot_A$ is monoidal with unit $A[X]$. An A-*plethory* is a monoid object in that monoidal category, that is, it is an A-A-biring P together with an associative map $\circ : P \odot_A P \longrightarrow P$ of A-A-birings (called *composition*) possessing a unit $e : A[X] \longrightarrow P$. (See any of [1, 2, 9] for details on these constructions.) An A-plethory is also known as an

A-A-biring monad object, an *A-A-biring triple*, or a *Tall-Wraith monad object* in the category of *A*-algebras. For any ring *A* the polynomial ring $A[X]$ has the structure of an *A*-plethory and in fact is an initial object in the category of *A*-plethories. In Section 4 we prove the following.

Proposition 1.5. *Let D be an integral domain. Any D-D-biring structure on* $\text{Int}(D)$ *compatible with that on $D[X]$ extends uniquely to a D-plethory structure on* $\text{Int}(D)$ *with unit given by the inclusion $D[X] \longrightarrow \text{Int}(D)$. Composition* $\circ : \text{Int}(D) \odot_D \text{Int}(D) \longrightarrow \text{Int}(D)$ *acts by ordinary composition on elements of the form $f \odot g$, that is, one has $\circ : f \odot g \longmapsto f \circ g$ for all $f, g \in \text{Int}(D)$.*

We also prove the following partial solution to Problem 1.2.

Theorem 1.6. *Let D be an integral domain.*

1. *Assume that there exists a D-D-biring structure on* $\text{Int}(D)$ *compatible with that on $D[X]$ and that the D-algebras $\text{Hom}_D(\text{Int}(D), A)$ and* $\text{Int}(D) \odot_D A$ *are D-torsion-free for any D-algebra A. Then the functors* $\text{Hom}_D(\text{Int}(D), -)$ *and $\text{Int}(D) \odot_D -$ are right and left adjoints, respectively, of the inclusion from D-torsion-free WPC algebras to D-algebras.*

2. *If D is a principal ideal domain (PID) with all residue fields finite, then the hypotheses and conclusion of statement (1) hold.*

In particular, any PID D with all residue fields finite satisfies the condition on D in Problem 1.2.

Note that if $\text{Int}(D) = D[X]$, which, for example, holds by [3, Corollary I.3.7] if D has no finite residue fields, then $\text{Hom}_D(\text{Int}(D), A)$ is naturally isomorphic to A and in particular is not D-torsion-free if A is not D-torsion-free. Of course in that case every D-algebra is WPC.

Finally, some remarks concerning the flatness of $\text{Int}(D)$. To contruct biring and plethory structures on $\text{Int}(D)$ it is convenient to assume that $\text{Int}(D)$ is flat over D. The hypothesis of the flatness of $\text{Int}(D)$ has already been shown useful for studying integer-valued polynomials rings. See [5, Theorems 3.6, 3.7, and 3.11], for example. In this paper, however, we really need only the *a priori* weaker hypothesis that the *n*-th tensor power $\text{Int}(D)^{\otimes n}$ of $\text{Int}(D)$ over D is D-torsion free for $n \leq 4$. Nevertheless, under certain general conditions, such as [5, Theorem 1.2 and Lemma 2.8], the domain $\text{Int}(D)$ is locally free, hence flat, as a D-module. This includes the case where D is a Krull domain or more generally a TV PVMD. Remarkably, however, there are no known examples of domains D such that $\text{Int}(D)$ is not free as a D-module, although we provide in [7, Section 2.3] some evidence for the following conjecture.

Conjecture 1.7. *The domain* $\text{Int}(D)$ *is not flat over D if $D = \mathbb{F}_2[[T^2, T^3]]$ or* $D = \mathbb{F}_2 + T\mathbb{F}_4[[T]]$*; in particular, there exists a local, Noetherian, one dimensional, analytically irreducible integral domain D such that $\text{Int}(D)$ is not flat over D.*

4

2 WPC algebras and tensor powers of $\mathrm{Int}(D)$

As in [5], we will say that a domain extension A, with quotient field L, of a domain D is *polynomially complete* if $\mathrm{Int}(D, A) = \mathrm{Int}(A)$, or equivalently if any polynomial $f \in L[X]$ such that $f(D) \subseteq A$ also satisfies $f(A) \subseteq A$. If D is not a finite field, then the domain $\mathrm{Int}(D^{\mathbf{X}})$ is the free polynomially complete extension of D generated by $\mathbf{X}$ [5, Proposition 2.4], and it is also the *polynomial completion of $D[\mathbf{X}]$ with respect to D* [5, Proposition 8.2].

As in [5, Section 7], and as in the introduction, we will say that a D-algebra A is *weakly polynomially complete*, or *WPC*, if for every $a \in A$ there exists a D-algebra homomorphism $\mathrm{Int}(D) \longrightarrow A$ sending X to a. Note that a D-torsion-free D-algebra A is WPC if and only if $f(A) \subseteq A$ for all $f \in \mathrm{Int}(D) \subseteq (A \otimes_D K)[X]$, where K is the quotient field of D. In particular, a domain extension A of D is WPC if and only if $\mathrm{Int}(D) \subseteq \mathrm{Int}(A)$. Any polynomially complete domain extension of a domain D is WPC, but the converse is not in general true, as revealed by the extension $\mathbb{Z}[T/2]$ of $\mathbb{Z}[T]$ [5, Example 7.3]. However, an extension A of D is polynomially complete if and only it is WPC, *provided that* $\mathrm{Int}(D, A)$ is equal to the A-algebra generated by $\mathrm{Int}(D)$.

For any set $\mathbf{X}$, the smallest subring of $\mathrm{Int}(D^{\mathbf{X}})$ containing $D[\mathbf{X}]$ that is closed under precomposition by elements of $\mathrm{Int}(D)$ is denoted $\mathrm{Int}_{\mathrm{w}}(D^{\mathbf{X}})$. For any domain D (finite or infinite), the domain $\mathrm{Int}_{\mathrm{w}}(D^{\mathbf{X}})$ is the free WPC extension of D generated by $\mathbf{X}$ [5, Proposition 7.2]. It is also the *weak polynomial completion $w_D(D[\mathbf{X}])$ of $D[\mathbf{X}]$ with respect to D*, as defined in [5, Section 8] and in Proposition 2.4 below.

In analogy with ordinary polynomial rings, there is for any set $\mathbf{X}$ a canonical D-algebra homomorphism

$$\theta_{\mathbf{X}} : \bigotimes_{X \in \mathbf{X}} \mathrm{Int}(D) \longrightarrow \mathrm{Int}(D^{\mathbf{X}}),$$

where the (possibly infinite) tensor product is over D and is a coproduct in the category of D-algebras. There are several large classes of domains for which $\theta_{\mathbf{X}}$ is an isomorphism for all $\mathbf{X}$, such as the Krull domains, the almost Newtonian domains [5, Section 5], and the PVMDs D such that $\mathrm{Int}(D_{\mathfrak{m}}) = \mathrm{Int}(D)_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ (hence the TV PVMDs as well by [6, Proposition 2.4]). However, we do not know whether or not $\theta_{\mathbf{X}}$ is an isomorphism for every domain D and every set $\mathbf{X}$. We say that the domain D is *polynomially composite* if $\theta_{\mathbf{X}}$ is an isomorphism for every set $\mathbf{X}$. Section 4 of [6] collects several known classes of polynomial composite domains. In Section 4, the polynomial compositeness of a domain D will be shown sufficient for a D-plethory structure to be defined on $\mathrm{Int}(D)$.

If $\mathrm{Int}_{\otimes}(D^{\mathbf{X}})$ denotes the image of $\theta_{\mathbf{X}}$, then we have $\mathrm{Int}_{\otimes}(D^{\mathbf{X}}) \subseteq \mathrm{Int}_{\mathrm{w}}(D^{\mathbf{X}})$, and equality holds for a given set $\mathbf{X}$ if and only if $\mathrm{Int}_{\otimes}(D^{\mathbf{X}})$ is a WPC extension of D. If equality holds for any set $\mathbf{X}$ then we will say that D is *weakly polynomially composite*.

Proposition 2.1. *The following conditions are equivalent for any integral domain D.*

1. *D is weakly polynomially composite.*

2. *$\mathrm{Int}_{\otimes}(D^{\mathbf{X}})$ is a WPC extension of D for any set $\mathbf{X}$.*

3. *$\mathrm{Int}_{\otimes}(D^n)$ is a WPC extension of D for some integer $n > 1$.*

4. *$\mathrm{Int}_{\otimes}(D^2)$ is a WPC extension of D.*

5. *For any element f of $\mathrm{Int}(D)$, the polynomials $f(X + Y)$ and $f(XY)$ lie in the image of $\theta_{\{X,Y\}}$.*

6. *The compositum of any collection of WPC D-algebras of D contained in some D-torsion-free D-algebra is again a WPC D-algebra.*

7. *The compositum of any collection of WPC extensions of D contained in some domain extension of D is again a WPC extension of D.*

Proof. Clearly we have $(1) \Leftrightarrow (2) \Rightarrow (3)$, $(4) \Rightarrow (5)$, and $(6) \Rightarrow (7) \Rightarrow (2)$. Thus we need only show that $(3) \Rightarrow (4)$ and $(5) \Rightarrow (6)$.

Suppose that statement (3) holds, and let $f(X, Y) \in \mathrm{Int}_{\otimes}(D^2)$. We may assume without loss of generality that the variables in $\mathrm{Int}(D^n)$ are $X, Y, X_3, X_4, \ldots, X_n$, whence $\mathrm{Int}(D^2) \subseteq \mathrm{Int}(D^n)$. It is then clear that $f(X, Y) \in \mathrm{Int}_{\otimes}(D^n)$. Now, let $g \in \mathrm{Int}(D)$. Then $g(f(X, Y))$ lies in $\mathrm{Int}_{\otimes}(D^n)$. Thus we can write

$$g(f(X, Y)) = \sum_i f_{i1}(X) f_{i2}(Y) f_{i3}(X_3) \cdots f_{in}(X_n),$$

where $f_{ij} \in \mathrm{Int}(D)$ for all i, j. Setting $X_i = 0$ for all $i > 2$, we see that

$$g(f(X, Y)) = \sum_i f_{i1}(X) f_{i2}(Y) f_{i3}(0) \cdots f_{in}(0),$$

whence $g(f(X, Y)) \in \mathrm{Int}_{\otimes}(D^2)$. Thus $\mathrm{Int}_{\otimes}(D^2)$ is a WPC extension of D. Therefore we have $(3) \Rightarrow (4)$.

Suppose that statement (5) holds. To prove (6), it suffices to show that the compositum C of two WPC D-algebras B and B' of D contained some a D-torsion-free D-algebra is again a WPC D-algebra. Let $f \in \mathrm{Int}(D)$, and let $b \in B$ and $b' \in B'$. By Proposition 2.1 each of the polynomials $f(X + Y)$ and $f(XY)$ can be written in the form $\sum_{i=1}^n g_i(X) h_i(Y)$, where the $g_i(X)$ and $h_i(Y)$ are integer-valued polynomials on D. It follows that $f(b + b')$ and $f(bb')$ lie in the the compositum C. Since this holds for all $b \in B$ and $b' \in B'$, it follows that $f(C) \subseteq C$. Therefore C is a WPC D-algebra. $\square$

Clearly polynomial compositeness implies weak polynomial compositeness. In fact, we have the following.

Proposition 2.2. *The following conditions are equivalent for any infinite integral domain D with quotient field K.*

(1) *$\theta_{\mathbf{X}}$ is surjective for every set $\mathbf{X}$.*

(2) *$\theta_{\mathbf{X}}$ is surjective for some infinite set $\mathbf{X}$.*

(3) *$\theta_{\mathbf{X}}$ is surjective for every finite set $\mathbf{X}$.*

(4) *$\mathrm{Int}(\mathrm{Int}(D^{\mathbf{X}}))$ is the $\mathrm{Int}(D^{\mathbf{X}})$-module generated by $\mathrm{Int}(D)$ for every (finite) set $\mathbf{X}$.*

(5) *$\mathrm{Int}_{\otimes}(D^{\mathbf{X}})$ is a polynomially complete extension of D for every (finite) set $\mathbf{X}$.*

(6) *One has $\mathrm{Int}_{\otimes}(D^{\mathbf{X}}) = \mathrm{Int}_{\mathrm{w}}(D^{\mathbf{X}}) = \mathrm{Int}(D^{\mathbf{X}})$ for every (finite) set $\mathbf{X}$.*

(7) *For any element f of $\mathrm{Int}(D)$, the polynomials $f(X+Y)$ and $f(XY)$ lie in the image of $\theta_{\{X,Y\}}$, and for any finite set $\mathbf{X}$ the domain $\mathrm{Int}(D^{\mathbf{X}})$ is the smallest subring of $K[\mathbf{X}]$ containing $D[\mathbf{X}]$ that is closed under precomposition by elements of $\mathrm{Int}(D)$.*

(8) *D is weakly polynomially composite, and every WPC domain extension of D is almost polynomially complete [5, Section 7].*

Proof. The first four conditions are equivalent by [5, Proposition 6.3]. Conditions (1) and (5) are equivalent because $\mathrm{Int}(D^{\mathbf{X}})$ is the polynomial completion of $D[\mathbf{X}]$ with respect to $\mathbf{X}$ [5, Example 8.3]. Conditions (1) and (6) are equivalent because $\mathrm{im}(\theta_{\mathbf{X}}) = \mathrm{Int}_{\otimes}(D^{\mathbf{X}}) \subseteq \mathrm{Int}_{\mathrm{w}}(D^{\mathbf{X}}) \subseteq \mathrm{Int}(D^{\mathbf{X}})$. Conditions (6) and (7) are equivalent by Proposition 2.1 and the definition of $\mathrm{Int}_{\mathrm{w}}(D^{\mathbf{X}})$. Finally, conditions (6) and (8) are equivalent by [5, Proposition 7.9]. $\qquad\square$

At the end of Section 8 of [5] it is noted how to construct the left adjoint of the inclusion functor from WPC domain extensions of D to domain extensions of D. The proof can be easily generalized to establish the following.

Proposition 2.3. *Let D be a domain with quotient field K, and let A be a D-torsion-free D-algebra.*

1. *A is contained in a smallest D-torsion-free WPC D-algebra, denoted $w_D(A)$, equal to the intersection of all WPC D-algebras containing A and contained in $A \otimes_D K$.*

2. *One has $w_D(A) = A$ if and only if A is WPC, and $w_D(A)$ is a domain if and only if A is a domain.*

3. *One has $w_D(A) \cong \mathrm{Int}_{\mathrm{w}}(D^{\mathbf{X}})/((\ker\varphi)K \cap \mathrm{Int}_{\mathrm{w}}(D^{\mathbf{X}}))$ for any surjective D-algebra homomorphism $\varphi : D[\mathbf{X}] \longrightarrow A$.*

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4. The association $A \longmapsto w_D(A)$ defines a functor from the category of D-torsion-free D-algebras to the category of D-torsion-free WPC D-algebras—both categories with morphisms as D-algebra homomorphisms—that is a left adjoint for the inclusion functor.

Assuming that D is weakly polynomially composite, we can also construct the right adjoint of the inclusion functor from D-torsion-free WPC D-algebras to D-torsion-free D-algebras.

Proposition 2.4. *Let D be a weakly polynomially composite domain, and let A be a D-torsion-free D-algebra.*

1. A contains a largest WPC D-algebra, denoted $w^D(A)$, equal to the compositum of all WPC D-algebras contained in A.

2. One has $w^D(A) = A$ if and only if A is WPC.

3. One has $w^D(A) = \{a \in A : a = \varphi(X) \text{ for some } \varphi \in \mathrm{Hom}_D(\mathrm{Int}(D), A)\}$.

4. The association $A \longmapsto w^D(A)$ defines a functor from the category of D-torsion-free D-algebras to the category of D-torsion-free WPC D-algebras—both categories with morphisms as D-algebra homomorphisms—that is a right adjoint for the inclusion functor.

Proof.

1. This follows from Proposition 2.1 and the fact that D itself is a WPC D-algebra.

2. This is clear from (1).

3. Let $a \in A$. Suppose that $a \in w^D(A)$. Then there is a D-algebra homomorphism $\psi : K[X] \longrightarrow K$ sending f to $f(a)$ for all $f \in K[X]$, where K is the quotient field of D, and ψ restricts to a D-algebra homomorphism $\varphi : \mathrm{Int}(D) \longrightarrow w^D(A) \subseteq A$ sending X to a. Conversely, suppose that there exists a D-algebra homomorphism $\varphi : \mathrm{Int}(D) \longrightarrow A$ sending X to a. Tensoring with K we see that φ is evaluation at a, that is, $\varphi(f) = f(a) \in A \otimes_D K$ for all $f \in \mathrm{Int}(D)$. Since $\mathrm{im}\,\varphi \subseteq A$ it follows that $f(a) \in A$ for all $f \in \mathrm{Int}(D)$. Thus we also have $g(\varphi(f)) = g(f(a)) = \varphi(g \circ f) \in A$ for all $f, g \in \mathrm{Int}(D)$. It follows that $\mathrm{im}\,\varphi \subseteq A$ is a WPC D-algebra and therefore $a \in \mathrm{im}\,\varphi \subseteq w^D(A)$.

4. Functoriality follows easily from (3). To prove adjointness, we must show that the natural map

$$\mathrm{Hom}_D(A, w^D(B)) \longrightarrow \mathrm{Hom}_D(A, B)$$

is a bijection for any D-torsion-free D-algebras A and B with A WPC. But this is clear from functoriality and (2).

$\square$

3   Biring structure on $\mathrm{Int}(D)$

The following result forms the basis for our investigation.

Theorem 3.1. *Let D be an integral domain.*

1. *If the domain $\mathrm{Int}(D)$ has a D-D-biring structure such that the inclusion $D[X] \longrightarrow \mathrm{Int}(D)$ is a homomorphism of D-D-birings, then D is weakly polynomially composite.*

2. *Assume that the n-th tensor power $\mathrm{Int}(D)^{\otimes n}$ of $\mathrm{Int}(D)$ over D is D-torsion-free for $n \leq 4$. Then $\mathrm{Int}(D)$ has a unique D-D-biring structure such that the inclusion $D[X] \longrightarrow \mathrm{Int}(D)$ is a homomorphism of D-D-birings if D is weakly polynomially composite.*

Proof. The usual D-D-biring co-operations on the polynomial ring $D[X]$ are given in [2, Example 1.2]. A compatible D-D-biring structure exists on $\mathrm{Int}(D)$ if and only if there exist D-algebra homomorphisms

$$
\begin{aligned}
\alpha : \quad & \mathrm{Int}(D) \quad \longrightarrow \mathrm{Int}(D) \otimes_D \mathrm{Int}(D) \\
o : \quad & \mathrm{Int}(D) \quad \longrightarrow D \\
\nu : \quad & \mathrm{Int}(D) \quad \longrightarrow \mathrm{Int}(D) \\
\mu : \quad & \mathrm{Int}(D) \quad \longrightarrow \mathrm{Int}(D) \otimes_D \mathrm{Int}(D) \\
\iota : \quad & \mathrm{Int}(D) \quad \longrightarrow D
\end{aligned}
$$

sending X, respectively, to $\alpha(X) = X \otimes 1 + 1 \otimes X$, $o(X) = 0$, $\nu(X) = -X$, $\mu(X) = X \otimes X$, and $\iota(X) = 1$, together satisfying the appropriate commutative diagrams, as well as a ring homomorphism

$$
\beta : D \longrightarrow \mathrm{Hom}_D(\mathrm{Int}(D), D)
$$

sending d to $f \longmapsto f(d)$ for all $d \in D$ and all $f \in D[X]$. (Such homomorphisms are, respectively, the co-addition, co-zero, co-additive inverse, co-multiplication, co-unit, and co-D-linear structure of a D-D-biring structure in $\mathrm{Int}(D)$ compatible with that of $D[X]$.)

Suppose that such D-algebra homomorphisms exist. Composing the homomorphisms α and μ with the homomorphism $\theta_{\{X,Y\}} : \mathrm{Int}(D) \otimes_D \mathrm{Int}(D) \longrightarrow \mathrm{Int}(D^2)$, we see that the polynomials $f(X+Y) = \theta_{\{X,Y\}}(\alpha(f))$ and $f(XY) = \theta_{\{X,Y\}}(\mu(f))$ lie in the image of $\theta_{\{X,Y\}}$ for every polynomial $f \in \mathrm{Int}(D)$. Condition (5) of Proposition 2.1 follows, and therefore D is weakly polynomially composite.

Suppose, conversely, that D is weakly polynomially composite and the n-th tensor power $\mathrm{Int}(D)^{\otimes n}$ of $\mathrm{Int}(D)$ over D is D-torsion-free for $n \leq 4$. Then the D-algebra $\mathrm{Int}(D) \otimes_D \mathrm{Int}(D)$ is isomorphic to $\mathrm{Int}_{\mathrm{w}}(D^2) = \mathrm{Int}_{\otimes}(D^2)$ and in particular is WPC domain extension of D. Therefore, for any $a \in \mathrm{Int}(D) \otimes_D \mathrm{Int}(D)$ there is a unique D-algebra homomorphism $\varphi : \mathrm{Int}(D) \longrightarrow \mathrm{Int}(D) \otimes_D \mathrm{Int}(D)$ sending X to a. The existence and uniqueness of the homomorphisms

α and μ thus follow. Likewise, homomorphisms o, ν, and ι also exist and are unique. Now, by hypothesis the D-algebra $\mathrm{Int}(D)^{\otimes_D n}$ injects into $\mathrm{Int}(D)^{\otimes_D n} \otimes_D K \cong K[X]^{\otimes_K n} \cong K[X_1, X_2, \ldots, X_n]$ for $n \leq 4$, where K is the quotient field of D. It follows that the usual K-$\mathbb{Z}$-biring co-operations on $K[X]$ restrict to the given co-operations on $\mathrm{Int}(D)$. Moreover, since all the commutative diagrams required of the co-operations on $K[X]$ to make $K[X]$ into a K-$\mathbb{Z}$-biring (as listed in [9, Appendix A], for example) only involve maps between the K-algebras $K[X]^{\otimes_K n}$ for $n \leq 4$, it follows that the same commutative diagrams hold for the co-operations on $\mathrm{Int}(D)$. Therefore, the restricted co-operations on $\mathrm{Int}(D)$ make $\mathrm{Int}(D)$ into a D-$\mathbb{Z}$-biring. Finally, one verifies that the map β exists, must act by $a \longmapsto (f \longmapsto f(a))$ for all $a \in A$ and all $f \in \mathrm{Int}(D)$, and is a homomorphism of rings. $\qquad\square$

Corollary 3.2. *If D is a polynomially composite domain, and in particular if D is a Krull domain or TV PVMD, then $\mathrm{Int}(D)$ has a unique D-D-biring structure such that the inclusion $D[X] \longrightarrow \mathrm{Int}(D)$ is a homomorphism of D-D-birings.*

We remark that there are no known examples of domains that are not polynomially composite, although Conjecture 1.7, if true, would yield two examples. We are thus led to the following problem.

Problem 3.3. *Determine whether or not the domain $D = \mathbb{F}_2[[T^2, T^3]]$ (resp., the domain $\mathbb{F}_2 + T\mathbb{F}_4[[T]]$) has a D-D-biring structure such that the inclusion $D[X] \longrightarrow \mathrm{Int}(D)$ is a homomorphism of D-D-birings.*

Remark 3.4.

1. *If all hypotheses in statements (1) and (2) of Theorem 3.1 hold, then for any $f \in \mathrm{Int}(D)$ and any $P \in D[\mathbf{X}]$, where $\mathbf{X} = \{X_1, \ldots, X_n\}$, one has $f(P) \in \mathrm{Int}_\otimes(D^{\mathbf{X}})$, so we may write*

$$f(P) = \sum_i \prod_{j=1}^n f_{ij}^P(X_j) \in \mathrm{Int}_\otimes(D^{\mathbf{X}}),$$

   where $f_{ij}^P \in \mathrm{Int}(D)$ for all i, j, in which case one has

$$P(\varphi_1, \varphi_2, \ldots, \varphi_n)(f) = \sum_i \prod_{j=1}^n \varphi_j(f_{ij}^P)$$

   for any $\varphi_1, \varphi_2, \ldots, \varphi_n \in \mathrm{Hom}_D(\mathrm{Int}(D), A)$.

2. *If only the hypothesis in statement (1) of Theorem 3.1 holds, then a similar conclusion of the above remark holds, where one instead uses the tensor product $\bigotimes_{k=1}^n \mathrm{Int}(D)$ over D (and the co-operations on $\mathrm{Int}(D)$) rather than the D-algebra $\mathrm{Int}_\otimes(D^n)$.*

By [4, Proposition 9.3] one has $\mathrm{Bin}(A) \cong \mathbb{Z}_p$ for any integral domain A of characteristic p, where $\mathbb{Z}_p$ denotes the ring of p-adic integers, and in particular one has $\mathrm{Bin}(\mathbb{F}_p) \cong \mathbb{Z}_p$. This generalizes as follows.

Proposition 3.5. *Let D be a Dedekind domain, and let $\mathfrak{m}$ be a maximal ideal of D with finite residue field. Then the map*

$$\widehat{D}_\mathfrak{m} \longrightarrow \operatorname{Hom}_D(\operatorname{Int}(D), D/\mathfrak{m})$$

acting by $\alpha \longmapsto (f \longmapsto f(\alpha) \bmod \mathfrak{m}\widehat{D}_\mathfrak{m})$ is a D-algebra isomorphism. More generally, for any domain extension A of D with $\mathfrak{m}A = 0$, the diagram

$$
\begin{array}{ccc}
\widehat{D}_\mathfrak{m} & \longrightarrow & \operatorname{Hom}_D(\operatorname{Int}(D), D/\mathfrak{m}) \\
& \searrow & \downarrow \\
& & \operatorname{Hom}_D(\operatorname{Int}(D), A)
\end{array}
$$

is a commutative diagram of D-algebra isomorphisms.

Proof. By [3, Theorem V.2.10], the prime ideals of $\operatorname{Int}(D)$ lying above $\mathfrak{m}$ are maximal and are in bijective correspondence with $\widehat{D}_\mathfrak{m}$, where $\alpha \in \widehat{D}_\mathfrak{m}$ corresponds to the maximal ideal

$$\mathfrak{m}_\alpha = \{f \in \operatorname{Int}(D) : \ f(\alpha) \in \mathfrak{m}\widehat{D}_\mathfrak{m}\}$$

of $\operatorname{Int}(D)$. Given any such α, the D-algebra homomorphism

$$\operatorname{eval}_\alpha : \operatorname{Int}(D) \longrightarrow \widehat{D}_\mathfrak{m}$$

acting by $\operatorname{eval}_\alpha : f \longrightarrow f(\alpha)$ induces D-algebra isomorphisms

$$\operatorname{Int}(D)/\mathfrak{m}_\alpha \cong \widehat{D}_\mathfrak{m}/\mathfrak{m}\widehat{D}_\mathfrak{m} \cong D/\mathfrak{m}.$$

It follows easily that the map $\widehat{D}_\mathfrak{m} \longrightarrow \operatorname{Hom}_D(\operatorname{Int}(D), D/\mathfrak{m})$ given in the statement of the proposition is a well-defined bijection. Moreover, this bijection is D-linear, and Remark 3.4 implies that it also preserves addition and multiplication (and unity) and is therefore an isomorphism of D-algebras. Finally, if A is any domain extension of D with $\mathfrak{m}A = 0$, then the kernel of any $\varphi \in \operatorname{Hom}_D(\operatorname{Int}(D), A)$ is a prime ideal of $\operatorname{Int}(D)$ lying over $\mathfrak{m}$ and therefore is of the form $\mathfrak{m}_\alpha$ for some $\alpha \in \widehat{D}_\mathfrak{m}$, whence φ factors through the map $\operatorname{eval}_\alpha$. It follows, then, that the D-algebra homomorphism $\operatorname{Hom}_D(\operatorname{Int}(D), D/\mathfrak{m}) \longrightarrow \operatorname{Hom}_D(\operatorname{Int}(D), A)$ is a bijection and therefore an isomorphism. $\square$

Using the full strength of [3, Theorem V.2.10], the result above can be generalized to any one-dimensional Noetherian analytically irreducible domain D such that $\operatorname{Int}(D)$ has a D-D-biring structure such that the inclusion $D[X] \longrightarrow \operatorname{Int}(D)$ is a homomorphism of D-D-birings.

4 Plethory structure on $\operatorname{Int}(D)$

As mentioned in the introduction, an A-plethory P, where A is a ring, is equipped with an associative operation $\circ : P \odot_A P \longrightarrow P$, called *composition*. The initial object in the category of A-plethories is the plethory $A[X]$.

Like $A[X]$, and in particular like the domain $D[X]$, the domain $\operatorname{Int}(D)$ has its own "internal" operation of composition. This leads to the following result.

Proposition 4.1. *Let D be an integral domain. Any D-D-biring structure on $\operatorname{Int}(D)$ such that the inclusion $D[X] \longrightarrow \operatorname{Int}(D)$ is a homomorphism of D-D-birings extends uniquely to a D-plethory structure on $\operatorname{Int}(D)$ with unit given by the inclusion $D[X] \longrightarrow \operatorname{Int}(D)$. Composition $\circ : \operatorname{Int}(D) \odot_D \operatorname{Int}(D) \longrightarrow \operatorname{Int}(D)$ acts by ordinary composition on elements of the form $f \odot g$, that is, one has $\circ : f \odot g \longmapsto f \circ g$ for all $f, g \in \operatorname{Int}(D)$.*

Proof. Let us first prove uniqueness. Since the inclusion $D[X] \longrightarrow \operatorname{Int}(D)$ is the unit one has $\circ(f \odot g) = f \circ g$ for all $f \in D[X]$ and all $g \in \operatorname{Int}(D)$. Therefore, for any $f \in \operatorname{Int}(D)$, choosing a nonzero $c \in D$ such that $cf \in D[X]$, we see that

$$c(\circ(f \odot g)) = \circ((cf) \odot g) = (cf) \circ g = c(f \circ g),$$

and therefore $\circ(f \odot g) = f \circ g$, for all $g \in \operatorname{Int}(D)$.

Now observe that, as with $D[X]$, ordinary composition on $\operatorname{Int}(D)$ satisfies the relations [2, 1.3.1–2], and the composition map $\circ$ on $\operatorname{Int}(D) \odot_D \operatorname{Int}(D)$ therefore exists. Associativity and the unit property are clear. However, one must also check that $\circ$ is a map of D-D-birings. To do so, it is necessary and sufficient, by Yoneda's lemma and the adjoint property of $\odot_D$, to verify that the map

$$\operatorname{Hom}_D(\operatorname{Int}(D), A) \longrightarrow \operatorname{Hom}_D(\operatorname{Int}(D), \operatorname{Hom}_D(\operatorname{Int}(D), A))$$

acting by

$$\varphi \longmapsto (f \longmapsto (g \longmapsto \varphi(g \circ f)))$$

is a (well-defined) D-algebra homomorphism for any D-algebra A. This is a straightforward but tedious computation that is left to the reader. $\square$

Corollary 4.2. *If D is a polynomially composite domain, and in particular if D is a Krull domain or TV PVMD, then $\operatorname{Int}(D)$ has a unique D-plethory structure with unit given by the inclusion $D[X] \longrightarrow \operatorname{Int}(D)$.*

Let A be a ring and P an A-plethory. A *P-ring* is an A-algebra R together with an A-algebra homomorphism $\circ : P \odot_A R \longrightarrow R$ such that $(\alpha \circ \beta) \circ r = \alpha \circ (\beta \circ r)$ and $e \circ r = e$ for all $\alpha, \beta \in P$ and all $r \in R$, where e is the image of X in the unit $A[X] \longrightarrow P$ [2, 1.9]. Such a map $\circ$ is said to be a *left action of P on R*. For example, P itself has a structure of a P-ring, as do the A-algebras $P \odot_A R$ and $\operatorname{Hom}_A(P, R)$ for any A-algebra R [2, 1.10], with left actions given by

$$\begin{aligned} P \odot_A (P \odot_A R) &\longrightarrow P \odot_A R \\ \alpha \odot (\beta \odot r) &\longmapsto (\alpha \circ \beta) \odot r \end{aligned}$$

and

$$\begin{aligned} P \odot_A \operatorname{Hom}_A(P, R) &\longrightarrow \operatorname{Hom}_A(P, R) \\ \alpha \odot \varphi &\longmapsto (\beta \longmapsto \varphi(\beta \circ \alpha)), \end{aligned}$$

respectively. Moreover, the functors $P \odot_A -$ and $W_P = \mathrm{Hom}_A(P, -)$ from A-algebras to P-rings are left and right adjoints, respectively, for the forgetful functor from P-rings to A-algebras [2, 1.10]. By Proposition 4.1, we therefore have the following.

Corollary 4.3. *Let D be an integral domain such that $\mathrm{Int}(D)$ has a D-plethory structure with unit given by the inclusion $D[X] \longrightarrow \mathrm{Int}(D)$. For any D-algebra A, the D-algebras $\mathrm{Int}(D) \odot_D A$ and $\mathrm{Hom}_D(\mathrm{Int}(D), A)$ both have $\mathrm{Int}(D)$-ring structures, with left actions given by*

$$
\begin{aligned}
\mathrm{Int}(D) \odot_D (\mathrm{Int}(D) \odot_D A) &\longrightarrow \mathrm{Int}(D) \odot_D A \\
f \odot (g \odot a) &\longmapsto (f \circ g) \odot a
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{Int}(D) \odot_D \mathrm{Hom}_D(\mathrm{Int}(D), A) &\longrightarrow \mathrm{Hom}_D(\mathrm{Int}(D), A) \\
f \odot \varphi &\longmapsto (g \longmapsto \varphi(g \circ f)),
\end{aligned}
$$

respectively. Moreover, the functors $\mathrm{Int}(D) \odot_D -$ and $W_{\mathrm{Int}(D)} = \mathrm{Hom}_D(\mathrm{Int}(D), -)$ from D-algebras to $\mathrm{Int}(D)$-rings are left and right adjoints, respectively, for the forgetful functor from $\mathrm{Int}(D)$-rings to D-algebras.

For any A-plethory P, the P-ring $W_P(R) = \mathrm{Hom}_A(P, R)$ of any A-algebra R is called the *P-Witt ring* of R. This terminology comes from the fact that, if P is the $\mathbb{Z}$-plethory Λ of [2, 2.11], then a P-ring is equivalently a λ-ring, and the functor W_P is isomorphic to the universal λ-ring functor Λ. If P is the $\mathbb{Z}$-plethory $\mathrm{Int}(\mathbb{Z})$, then a P-ring is equivalently a binomial ring, and the functor W_P is isomorphic to the functor Bin. The latter fact generalizes to the following result, which implies Theorem 1.6 of the introduction.

Theorem 4.4. *Let D be an integral domain with quotient field K such that $\mathrm{Int}(D)$ has a D-plethory structure with unit given by the inclusion $D[X] \longrightarrow \mathrm{Int}(D)$, and let A be a D-algebra.*

1. *If there exists an $\mathrm{Int}(D)$-ring structure on A, then A is WPC.*

2. *If A is D-torsion-free, then there exists a (necessarily unique) $\mathrm{Int}(D)$-ring structure on A if and only if A is WPC.*

3. *If A is D-torsion-free, then the D-algebra homomorphism $W_{\mathrm{Int}(D)}(A) \longrightarrow A$ is an inclusion with image equal to $w^D(A)$, and the functor w^D is therefore isomorphic to the functor $W_{\mathrm{Int}(D)}$ restricted to the category of D-torsion-free D-algebras.*

4. *If A is D-torsion-free, then the D-algebra homomorphism $\mathrm{Int}(D) \odot_D A \longrightarrow A \otimes_D K$ acting by $f \odot a \longmapsto f(a)$ has image equal to $w_D(A)$, and the functor w_D is therefore isomorphic to the functor $T \circ (\mathrm{Int}(D) \odot_D -)$ restricted to the category of D-torsion-free D-algebras, where $T(B)$ for any D-algebra B denotes the image of B in $B \otimes_D K$.*

5. If A is D-torsion-free and WPC, then the natural D-algebra homomorphisms $W_{\mathrm{Int}(D)}(A) \longrightarrow A$ and $A \longrightarrow T(\mathrm{Int}(D) \odot_D A)$ are isomorphisms.

6. The functor $T \circ (\mathrm{Int}(D) \odot_D -)$ is a left adjoint for the inclusion from D-torsion-free WPC D-algebras to D-algebras.

7. The functors $W_{\mathrm{Int}(D)}$ and $\mathrm{Int}(D) \odot_D -$ are right and left adjoints, respectively, for the inclusion from D-torsion-free WPC D-algebras to D-algebras if and only if the $\mathrm{Int}(D)$-rings $W_{\mathrm{Int}(D)}(R)$ and $\mathrm{Int}(D) \odot_D R$ are D-torsion-free for every D-algebra R.

8. Every $\mathrm{Int}(D)$-ring is D-torsion-free if D is a PID with all residue fields finite.

Proof.

1. Let $a \in A$. Define $\varphi : \mathrm{Int}(D) \longrightarrow A$ by $\varphi(f) = f \circ a$ for all $f \in \mathrm{Int}(D)$. Then φ is a ring homomorphism with $\varphi(X) = a$.

2. Let A be a D-torsion-free D-algebra. First, suppose there exists an $\mathrm{Int}(D)$-ring structure on A. Let $f \in \mathrm{Int}(D)$. If $f \in D[X]$ then one has $f \circ a = f(a)$. But in any case there exists a nonzero $c \in D$ such that $cf \in D[X]$, and then one has

$$c(f \circ a) = (c \circ a)(f \circ a) = (cf) \circ a = (cf)(a) = cf(a),$$

whence $f \circ a = f(a)$. Thus an $\mathrm{Int}(D)$-ring structure on A, if it exists, is unique. Suppose, then, that A is a WPC D-algebra. Let F be the free D-algebra generated by expressions of the form $f \odot a$ for $f \in \mathrm{Int}(D)$ and $a \in A$. The D-algebra homomorphism $F \longrightarrow A$ acting by $f \odot a \longmapsto f(a)$ respects the relations [2, 1.3.1–2] and therefore factors through a D-algebra homomorphism $\mathrm{Int}(D) \odot_D A \longrightarrow A$. Finally, since $(f \circ g) \circ a = f \circ (g \circ a)$ and $X \circ a = a$ for all $f, g \in \mathrm{Int}(D)$ and $a \in A$, it follows that A has the structure of an $\mathrm{Int}(D)$-ring.

3. This follows from Proposition 2.3.

4. This follows from Proposition 2.4.

5. This follows from statements (3) and (4).

6. Let A be a D-algebra and B a D-torsion-free WPC D-algebra, and let P denote the plethory $\mathrm{Int}(D)$. Then we have canonical bijections

$$\begin{aligned} \mathrm{Hom}_D(T(P \odot_D A), B) &\cong \mathrm{Hom}_D(P \odot_D A, B) \\ &\cong \mathrm{Hom}_D(A, W_P(B)) \\ &\cong \mathrm{Hom}_D(A, B). \end{aligned}$$

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7. Necessity is clear. To prove sufficiency note that, with notation as in (6), we have canonical bijections

$$\begin{aligned} \operatorname{Hom}_D(B, W_P(A)) &\cong \operatorname{Hom}_D(P \odot_D B, A) \\ &\cong \operatorname{Hom}_D(T(P \odot_D B), A) \\ &\cong \operatorname{Hom}_D(B, A). \end{aligned}$$

8. Let D be a PID with all residue fields finite, and let A be an $\operatorname{Int}(D)$-ring. To show that A is D-torsion-free, it suffices to show that A has no π-torsion for any prime $\pi \in D$. Now, $D/(\pi)$ is by hypothesis a finite field, say, having q elements. The polynomial $F = (X^q - X)/\pi$ is then an element of $\operatorname{Int}(D)$. Note that $f(X, Y) = F(X + Y) - F(X) - F(Y)$ lies in $(X, Y)D[X, Y]$, and therefore

$$F \circ 0 = F \circ (0 + 0) = F \circ 0 + F \circ 0 + f(0, 0) = F \circ 0 + F \circ 0$$

in A, and therefore $F \circ 0 = 0$. Note also that $F(\pi X) = \pi^{q-1} X^q - X$. Therefore, if $\pi b = 0$, then

$$0 = F \circ (\pi b) = F \circ ((\pi X) \circ b) = F(\pi X) \circ b = \pi^{q-1} b^q - b = -b,$$

whence $b = 0$.

$\square$

We end with the following problem.

Problem 4.5. *Determine equivalent conditions on an integral domain D so that the D-algebra $\operatorname{Int}(D)$ has a D-plethory structure with unit given by the inclusion $D[X] \longrightarrow \operatorname{Int}(D)$ and so that the D-algebras $W_{\operatorname{Int}(D)}(R)$ and $\operatorname{Int}(D) \odot_D R$ are D-torsion-free for every D-algebra R.*

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